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On the H^2 Two-Side Model Matching Problem with Preview

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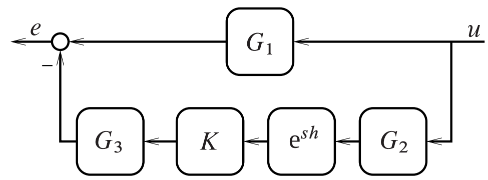
***Abstract*— The H^2 optimization problem with preview and asymptotic behavior constraints is considered in a general two-side model matching setting. The solution is obtained in terms of two constrained Sylvester equations, associated with asymptotic behavior, and stabilizing solutions of two algebraic Riccati equations. The Riccati equations do not depend on the preview length, yet are affected by asymptotic behavior constraints and are thus different from the standard H^2 Riccati equations arising in problems with no steady-state requirements or in one-side problems.**

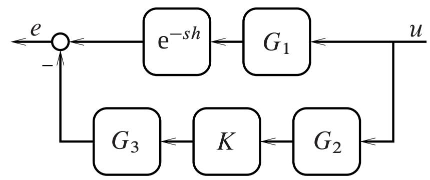
I. Introduction and Problem Formulation

Numerous estimation and control problems fall into the category of problems with information preview. For example, in many communication and signal processing applications, certain delay between estimation generation and estimated signal is tolerable. In such problems the allowed delay may be interpreted as availability of future measurements within a constant preview window. Also in some tracking and disturbance attenuation problems, e.g., those arising in robotics or active suspension control, preview of command and/or disturbance signals may be available to a controller. Clearly, availability of preview can potentially improve the performance of the controller or the estimator. In this work, the question of how to exploit this potential will be addressed in the framework of a general H^2 model matching optimization.

Both estimation and open-loop control problems with information preview can be cast as a unified setting referred to as model matching with preview. Moreover, many closed-loop control problems with preview can be also reduced to this setting using the Youla-Kucera parameterization [6], [20]. Two different representations of model matching with preview are depicted in Fig. 1. In these block diagrams, G_1, G_2 and G_3 represent some given causal LTI systems, containing the problem data and (possibly unstable) weights. The design parameter K represents an estimator / controller / Youla parameter. The block diagram depicted in Fig. 1(a) is natural in control applications, where the reference and/or the measurements of the disturbance signals are available to the controller in advance. The block diagram in Fig. 1(b) is relevant for estimation problems, where preview is available due to the latency allowed in estimation generation. It is readily seen that these two settings are equivalent up to a shift of the time axes. For the sake of convenience and without loss of the generality, in this work we will adopt the setting depicted on Fig. 1(b), where the availability of preview is reflected by the causal delay element e^{-sh}. In particular, the causality of the overall system in this case renders its stability equivalent to the condition $T \in H^\infty$.



(a) Control setting



(b) Estimation setting

Fig. 1. Model matching with preview

The one-side version of model matching with preview, i.e., the setting with either $G_2 = I$ or $G_3 = I$ is currently well studied in both H^2 and H^∞ settings, [11]–[13]. Yet, many problems of interest, such as measured disturbance attenuation, etc., have an intrinsic two-side structure. This motivates our study of the general two-side model matching with preview, which, to the best of our knowledge, has not been addressed in the literature yet. In this work we focus on the H^2 optimization problem, which accounts for both transient and asymptotic behavior of the underlying control/estimation system and can be formulated as follows.

OP: Given proper rational transfer matrices G_1, G_2, G_3 and a constant $h \geq 0$, find $K \in H^\infty$, which guarantees

$$T = e^{-sh} G_1 - G_3 K G_2 \in H^2 \cap H^\infty \qquad (1)$$

and minimizes $\|T\|_2$.

The one-side version of this problem with $G_3 = I$ will hereafter be referred to as one-side problem and will be denoted by **OPo**.

The following remarks may be useful for understanding and interpreting the problems formulated above.

1) The transfer matrices G_1, G_2 and G_3 may not belong to H^∞ due to unstable weights, used for description of the underlying control/estimation system. In this context, the stability constraint $T \in H^\infty$ in (1) accounts for asymptotic behavior of the system subject to non-decaying input signals, modeled by the weights instabilities. More details on casting asymptotic behavior requirements as input-output stabilization problem can be found in [3], [5].

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2) The optimization criterion, $\|T\|_2$, accounts for transient behavior of the underlying system.
3) In the case when the considered model matching setting originates from an open-loop control or an estimation problem, the design parameter K has to be stable and causal in order to guarantee implementability of the resulting solution. In this context, as well as in case when K represents a Youla parameter, the requirement $K \in H^\infty$ is natural.
4) The constant h represents the length of preview available in the problem. One way to verify this is by noting that multiplication of T by e^{sh} does not change its L^2 norm and, as a result, the optimization criterion can be rewritten as
$$\|T\|_2 = \|\mathrm{e}^{sh}T\|_2 = \|G_1 - G_3 K \mathrm{e}^{sh} G_2\|_2.$$
Thus, shifting the time axes, the presence of the delay element in (1) can be interpreted as the presence of a preview in the input of K. At this point, we can redefine the design parameter as $\tilde{K} := K\mathrm{e}^{sh} \in \mathrm{e}^{sh}H^\infty$ and rewrite the optimization criteria as $\|G_1 - G_3\tilde{K}G_2\|_2$. In this representation, the increase of h can be interpreted as the relaxation of the causality requirement on $\tilde{K}$ and infinite preview length corresponds to the problem without causality constraints on the design parameter.

Studies of H^2 problems related to **OP** in the context of preview control go back to the '60s, [1], [17], [18]. In estimation framework, the interest to problems with information preview, referred to as smoothing, can be traced even to earlier decades [19]. In the late '80s and mid '90s, several results in the discrete-time setting were published, see [14], [15], and over the past few years the area of continuous-time preview control gained a renewed interest. The two-block H^2 preview tracking was addressed in the 2DOF setting in [10], [16] and later on as a feedforward tracking in [2], [9], [13]. In [13] also the multiple preview case was addressed. The problems considered in the references above, however, correspond to the one-side setting with no stability constraints. A problem with stability constraints was considered in the context of H^∞ optimization in [11], [12]. Yet, also in these works the discussion and, as a result, the stabilization procedure are restricted to the one-side case. A very general stabilization problem corresponding to two-side setting was considered in [7], where a complete but rather complicate parameterization of all stabilizing solutions was derived in terms of three independent parameters. This result was exploited in [8] to solve general finite-dimensional H^2 optimization problem with stability constraints. The solution method proposed in this work, however, is not readily extendible to the problem with information preview.

Solution of the one-side problem, **OPo**, can be derived by combining stabilization procedure from [11], [12] and classical L^2 optimization arguments. Yet, the extension of this method to the general two-side setting is by no means trivial and constitutes the main challenge of the current work. The difficulties arise already in the stabilization stage, where the general problem exhibits a more complicate nature then its one-side counterpart. Recently, it was shown, see [3], [5], that under mild simplifying assumptions, all stabilizing solutions of the general model matching problem can be characterized by an affine parameterizations in terms of a single stable but otherwise arbitrary parameter. This result turns out to be highly relevant in the context of **OP** and serves as a starting point for the current study.

In this work we reconsider the stabilization procedure from [3], [5] and rewrite the parameterization of all stabilizing solutions in a form, suitable for the treatment of the optimization problem. This enables us to extend the existing methods of the solution of the one-side problem to the two-side setting. We end up with an explicit and numerically efficient solution, which provides insight into the structure of the resulting controller/estimator. The solution is given in terms of two matrix Sylvester equations, associated with stabilization, and two algebraic Riccati equations. The latter equations differ from the standard H^2 Riccati equations by a shift of A-matrices, which relies on solutions of the aforementioned Sylvester equations.

We will consider **OP** under the following set of assumptions:

$\boldsymbol{\mathcal{A}_1}$: $G_1(\infty) = 0$,

$\boldsymbol{\mathcal{A}_2}$: $G_2(\infty)$ and $G_3(\infty)$ have full row and column rank respectively,

$\boldsymbol{\mathcal{A}_3}$: $(\mathfrak{Z}_2 \cup \mathfrak{Z}_3) \cap \mathrm{j}\mathbb{R} = \emptyset$,

$\boldsymbol{\mathcal{A}_4}$: $(\mathfrak{P}_2 \cup \mathfrak{P}_3) \cap \mathbb{C}^+ \in \mathrm{j}\mathbb{R}$,

where $\mathfrak{Z}_{2/3}$ and $\mathfrak{P}_{2/3}$ refer to sets of all zeros and poles of $G_{2/3}$ respectively. The first assumption is technical and imposes no loss of the generality. Indeed, **OP** is solvable only if there exists K that renders $T \in H^2$ and, in particular, guarantees that $T(\infty) = 0$. This, in turn, is possible only if there exists a matrix D_k such that $G_1(\infty) - G_3(\infty)D_k G_2(\infty) = 0$. In this case, the design parameter can be shifted as $\tilde{K} = K - \mathrm{e}^{-sh}D_k$ to yield a problem in which $\boldsymbol{\mathcal{A}_1}$ is satisfied. The assumptions $\boldsymbol{\mathcal{A}_2}$ and $\boldsymbol{\mathcal{A}_3}$ are standard and rule out problem redundancy and singularity. The fourth assumption is practically not restrictive, since, typically, unstable poles present in **OP** originate from unstable weights with imaginary axis instabilities. This assumption rules out possibility of the coincidence between unstable poles and zeros of $G_{2/3}$ and facilitates the stabilization procedure, see [3], [5] for more details.

The paper is organized as follows. In Section II the rationale behind the proposed solution is described in frequency domain. In Section III explicit formulae of **OP** solution are derived using the state-space machinery. Finally, some concluding remarks are available in Section IV.

Notation: The open left and right halves of the complex plain are denoted by $\mathbb{C}^-$ and $\mathbb{C}^+$ respectively. For any left-invertible $A \in \mathbb{R}^{n\times m}$, the matrices $A^+ \in \mathbb{R}^{m\times n}$ and $A^\perp \in \mathbb{R}^{n-m\times n}$ denote a pseudo inverse of A and its complement satisfying
$$\begin{bmatrix} A^+ \\ A^\perp \end{bmatrix} A = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \det \begin{bmatrix} A^+ \\ A^\perp \end{bmatrix} \neq 0.$$

Similarly, if A is right-invertible, $A^+ \in \mathbb{R}^{m\times n}$ and $A^\perp \in \mathbb{R}^{m\times m-n}$ satisfy

$$A\begin{bmatrix} A^+ & A^\perp \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \quad \det\begin{bmatrix} A^+ & A^\perp \end{bmatrix} \neq 0.$$

Given a transfer matrix $G(s)$, its pseudo inverse is denoted by $G^\#$ and its conjugate is denotes $G^\sim(s)$ and defined as $G^\sim(s) = [G(-s)]'$. For $G \in L^2$, $(G)_+$ refers to the projection of G on $H^2_\perp$. For any rational strictly proper transfer function given by its minimal state-space realization $G(s) = C(sI-A)^{-1}B$, the completion operator constitutes an FIR linear system and is defined as

$$\pi_h\{G(s)\} = \left[\begin{array}{c|c} A & B \\ \hline Ce^{-Ah} & 0 \end{array}\right] - \mathrm{e}^{-sh}\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$

More details on this definition can be found in [11]. The left and the right coprime factorizations are abbreviated as lcf and rcf, respectively. Doubly coprime factorizations for each of the transfer matrices involved in **OP** are denoted by

$$G_i = N_i M_i^{-1} = \tilde{M}_i^{-1}\tilde{N}_i, \tag{2a}$$

$$\begin{bmatrix} X_i & Y_i \\ -\tilde{N}_i & \tilde{M}_i \end{bmatrix}\begin{bmatrix} M_i & -\tilde{Y}_i \\ N_i & \tilde{X}_i \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{2b}$$

for $i = 1,2,3$. The sets of all poles and zeros of $G_i(s)$ are denoted by $\mathfrak{P}_i$ and $\mathfrak{Z}_i$, respectively. The notion of *bilateral Diophantine equation* (BDE) on rational matrices refers to the equation of the form $MX + YN = P$, where $M, N, P \in H^\infty$ are given and $X, Y \in H^\infty$ are to be found.

II. Frequency domain solution

In this section a frequency domain solution of **OP** is derived. The results presented here are not readily computable, the aim of this section is to explain the main steps of the solution, which later on will be implemented using the state-space techniques to get the final result.

Our first step will be to resolve the stability constraint imposed on T and to reduce the problem to the L^2 optimization. In the one-side case, this can be done using the parameterization of all stabilizing solutions available from [11], [12]. Although the two-side stabilization problem is of a more complicate nature, it turns out, [3], [5], that under assumptions $\mathcal{A}_4$, $\mathcal{A}_3$ similar—in terms of its structure—parameterization exists also in two-side case. The following result can be formulated.

Lemma 1: Let $\mathcal{A}_{2\text{-}4}$ hold. Then there exists $K \in H^\infty$ that stabilize (1) iff $\tilde{M}_3 G_1 M_2 \in H^\infty$ and there exist U_2, V_2, U_3, and V_3 in H^∞ that satisfy the following BDE's:

$$N_3U_2 + V_2\tilde{M}_2 = \tilde{X}_3\tilde{M}_3G_1M_2Y_2, \tag{3}$$

$$U_3\tilde{N}_2 + M_3V_3 = \tilde{Y}_3\tilde{M}_3G_1M_2X_2. \tag{4}$$

If these conditions are satisfied, then all $K \in H^\infty$ that stabilize (1) and all the corresponding stabilized T's can be characterized as

$$K = \mathrm{e}^{-sh}\bar{K}_1 + \bar{K}_3Q\bar{K}_2, \tag{5}$$

$$T = \mathrm{e}^{-sh}\bar{G}_1 - \bar{G}_3Q\bar{G}_2, \tag{6}$$

where

$$\begin{bmatrix} \bar{K}_1 & \bar{K}_3 \\ \bar{K}_2 & \end{bmatrix} := \begin{bmatrix} \tilde{Y}_3\tilde{M}_3G_1M_2Y_2 + M_3U_2 + U_3\tilde{M}_2 & M_3 \\ \tilde{M}_2 & \end{bmatrix},$$

$$\begin{bmatrix} \bar{G}_1 & \bar{G}_3 \\ \bar{G}_2 & \end{bmatrix} := \begin{bmatrix} \tilde{X}_3\tilde{M}_3G_1M_2X_2 + V_2\tilde{N}_2 + N_3V_3 & N_3 \\ \tilde{N}_2 & \end{bmatrix},$$

and $Q \in H^\infty$ but otherwise arbitrary.

Proof: Using the finite-dimensional result of [5], it can be shown that any K given by (5) stabilizes the problem and that the corresponding stabilized T is given by (6). The only thing left is to show completeness this parameterization. To this end, using completion operator, we may rewrite the expression for T as follows

$$T = -\pi_h\{G_1\} + \tilde{G}_1 - G_3KG_2,$$

where $\tilde{G}_1$ is finite dimensional and the first term is a stable FIR system, which does not affect the stabilization. This reduces the original stabilization problem to a similar but finite-dimensional one, which was considered in [5]. The proof can be completed now by noting that, according to [5], affine parameterization of a form (5) is complete for any finite-dimensional two-side problem that satisfies $\mathcal{A}_{2\text{-}4}$. ■

Following the ideas from [12], we will assume without loss of the generality that $\bar{G}_1$, $\bar{G}_2$ and $\bar{G}_3$ involved in the parameterization of all stable T's in (6) satisfy

$$\bar{G}_2\bar{G}_2^\sim = I, \quad \bar{G}_3^\sim\bar{G}_3 = I, \quad \bar{G}_3^\sim\bar{G}_1\bar{G}_2^\sim \in H^2_\perp. \tag{7}$$

It will be shown in the next section that such parameterization can alway be constructed. The properties in (7) facilitate the derivation of compact formulae for the frequency domain solution and greatly simplify state-space derivations in the next section.

The result of Lemma 1 involves bilateral Diophantine equations[1], which renders it more complicated then its one-side counterpart from [11], [12]. Still, similarly to the one-side case, it yields an affine parameterization of all stabilizing solutions given in terms of single parameter. As a result, reformulating **OP** in terms of a new parameter Q, we can resolve stability constraints without changing the structure of the optimization criteria. Namely, we can reduce the problem to finding $Q \in H^\infty$ that minimizes the L^2 norm of the expression given in (6). Note that under assumptions $\mathcal{A}_2$ and $\mathcal{A}_3$ there exist $\bar{G}_2^\#, \bar{G}_3^\# \in L^\infty$. Post- and pre-multiplying (6) by these transfer functions yields

$$Q = \mathrm{e}^{-sh}\bar{G}_3^\#\bar{G}_1\bar{G}_2^\# - \bar{G}_3^\# T\bar{G}_2^\#,$$

showing that $T \in L^2$ only if $Q \in L^2$ and implying that the domain of the optimization parameter can be effectively replaced by $Q \in H^2$.

Thus, **OP** reduces to the L^2 optimization

$$Q^{\mathrm{opt}} = \underset{Q\in H^2}{\operatorname{argmin}} \|\mathrm{e}^{-sh}\bar{G}_1 - \bar{G}_3Q\bar{G}_2\|_2, \tag{8}$$

which can be solved using the standard Hilbert space arguments. A geometric interpretation of this problem is presented

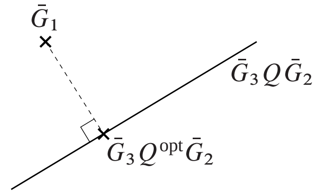

Fig. 2. Geometric interpretation of (8)

in Fig. 2. By the Projection Theorem, the optimal solution can be characterized by the orthogonality of the optimal error to the subspace generated by $\bar{G}_3 Q \bar{G}_2$, i.e., by

$$\langle {\rm e}^{-sh} \bar{G}_1 - \bar{G}_3 Q^{\rm opt} \bar{G}_2, \bar{G}_3 Q \bar{G}_2 \rangle_2 = 0, \quad \forall Q \in H^2.$$

Because the transfer function of the adjoint operator is the conjugate transfer function, the condition above reads

$$\langle \bar{G}_3^{\sim} ({\rm e}^{-sh} \bar{G}_1 - \bar{G}_3 Q^{\rm opt} \bar{G}_2) \bar{G}_2^{\sim}, Q \rangle_2 = 0, \quad \forall Q \in H^2.$$

Using (7), we then obtain that the optimal Q must satisfy

$${\rm e}^{-sh} \bar{G}_3^{\sim} \bar{G}_1 \bar{G}_2^{\sim} - Q^{\rm opt} \in H^2_{\perp},$$

which, in turn, yields

$$Q^{\rm opt} = \left({\rm e}^{-sh} \bar{G}_3^{\sim} \bar{G}_1 \bar{G}_2^{\sim} \right)_+ . \tag{9}$$

The expression for the minimal norm of the error can be given by

$$\|T^{\rm opt}\|_2^2 = \|\bar{G}_1\|_2^2 - \|\bar{G}_3 Q^{\rm opt} \bar{G}_2\|_2^2 = \|\bar{G}_1\|_2^2 - \|Q^{\rm opt}\|_2^2,$$

where we made use of the facts that $\bar{G}_2$ and $\bar{G}_3$ are co-inner and inner, respectively (cf. (7)).

Going back to the original problem, i.e., substituting (9) into the parameterizations from Lemma 1, yields the main result of the current section.

Theorem 1: Let the conditions of Lemma 1 hold and assume that parameterizations (5) and (6), are constructed to satisfy (7). Then, **OP** is solvable and the optimal K with the corresponding minimal $\|T\|^2$ are given by

$$K^{\rm opt} = {\rm e}^{-sh} \bar{K}_1 + \bar{K}_3 ({\rm e}^{-sh} \bar{G}_3^{\sim} \bar{G}_1 \bar{G}_2^{\sim})_+ \bar{K}_2, \tag{10}$$

$$\|T^{\rm opt}\|_2^2 = \|\bar{G}_1\|_2^2 - \|\left({\rm e}^{-sh} \bar{G}_3^{\sim} \bar{G}_1 \bar{G}_2^{\sim} \right)_+ \|_2^2. \tag{11}$$

Note that, according to (7), the system $\bar{G}_3^{\sim} \bar{G}_1 \bar{G}_2^{\sim}$ is anticausal. Therefore, $F_{\rm FIR} := ({\rm e}^{-sh} \bar{G}_3^{\sim} \bar{G}_1 \bar{G}_2^{\sim})_+$ is an FIR system of the form

$$F_{\rm FIR} = \pi_h \{ \bar{G}_3^{\sim} \bar{G}_1 \bar{G}_2^{\sim} \}.$$

The structure of the optimal solution is presented in Fig. 3 and has a clear interpretation. The presence of the delay element implies that the upper branch of the diagram does not make use of the information preview and acts as if no preview was available in the problem. The system $\bar{K}_1$ does not depend on the preview length h and coincides with the optimal solution of the problem without preview. On the other hand, the lower branch of the diagram, based on the FIR block, accounts for the existence of preview and makes the difference between the solutions of **OP** and the standard preview-free optimization.

[1] See [3], [4] to find more details on the bilateral Diophantine equation over $\mathcal{RH}_\infty$ and its relation to stabilization problem.

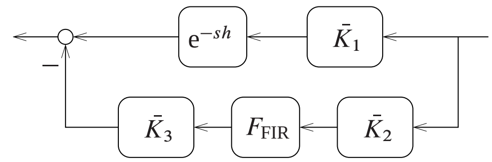

Fig. 3. Optimal solution structure

III. State-space solution

Having described the main steps of the solution in the frequency domain, we aim at implementing them using state-space techniques in order to derive computable formulae for the optimal solution and for the achievable performance.

A. The one-side problem

As a preliminary step before considering the general two-side problem, let us focus on its one-side version, **OPo**, with

$$T = {\rm e}^{-sh} G_1 - K G_2. \tag{12}$$

In this case we can choose $\tilde{M}_3$, $\tilde{N}_3$, M_3, and $\tilde{Y}_3$ as identity matrices and $\tilde{X}_3 = 0$. This renders BDE's (3) and (4) trivial, with possible solutions

$$U_2 = 0, \quad V_2 = 0, \quad U_3 = 0, \quad \text{and} \quad V_3 = G_1 M_2 X_2.$$

This simplifies the solution **OPo** considerably.

Consider the following composite system, given by its minimal state-space realization

$$G := \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_1 & 0 \\ C_2 & D_2 \end{array} \right]. \tag{13}$$

To simplify the exposition, assume hereafter that

$\boldsymbol{\mathcal{A}}_5$: $D_2 D_2' = I$,

which is a matter of scaling and can thus be assumed without loss of the generality provided $\boldsymbol{\mathcal{A}}_2$ holds true. Following the solution line described in Section II, our first step is to reduce the problem to the L^2 optimization. The result below is essentially from [11], [12].

Lemma 2: Let G be given by its minimal state-space realization (13). Then there exists $K \in H^\infty$ stabilizing (12) iff (A, C_2) is detectable. If this condition holds and L is chosen so that $A + LC_2$ is Hurwitz, then all $K \in H^\infty$ stabilizing (12) and all the corresponding stabilized T's can be characterized by (5) and (6) respectively, where $\bar{K}_3 = I$, $\bar{G}_3 = I$,

$$\begin{bmatrix} \bar{K}_1 \\ \bar{K}_2 \end{bmatrix} = \left[\begin{array}{c|c} A_L & L \\ \hline -C_1 & 0 \\ C_2 & {\rm I} \end{array} \right], \tag{14}$$

$$\bar{G} := \begin{bmatrix} \bar{G}_1 \\ \bar{G}_2 \end{bmatrix} = \left[\begin{array}{c|c} A_L & B + LD_2 \\ \hline C_1 & 0 \\ C_2 & D_2 \end{array} \right], \tag{15}$$

and $A_L = A + LC_2$ and $Q \in H^\infty$ but otherwise arbitrary.

The lemma above yields a state-space representation of the one-side version of Lemma 1, which enables one to replace the original model-matching setup with a stabilized one, associated with $\bar{G}_1$ and $\bar{G}_2$. It is worth mentioning that the order of the state-space realization (15), which corresponds to the stabilized model-matching setup, equals to the order of the original system G in (13). This shows that the stability constraints of **OPo** can be resolved without increasing complexity of the problem.

The systems $\bar{K}_1$, $\bar{K}_2$, $\bar{G}_1$ and $\bar{G}_2$, given in (14), (15), are not unique due to the freedom in the choice of L. This freedom can be exploited to get parameterizations that satisfy (7). To this end, choose

$$L = -YC_2' - BD_2', \tag{16}$$

where $Y \geq 0$ is the stabilizing solution of the following algebraic Riccati equation (ARE):

$$AY + YA' - (YC_2' + BD_2')(C_2Y + D_2B') + BB' = 0. \tag{17}$$

The existence of a unique stabilizing solution of this equation for any stabilizable problem is guaranteed by $\boldsymbol{\mathcal{A}_3}$ and the detectability of (A, C_2), see [21]. It can now be verified by straightforward calculations that in this case (7) hold and

$$\bar{G}_1\bar{G}_2^{\sim} = \left[\begin{array}{c|c} -A_L' & C_2' \\ \hline -C_1Y & 0 \end{array}\right]. \tag{18}$$

Substituting (14) and (18) into (10) and (11) leads to the following explicit solution of **OPo**:

Theorem 2: Let G be given by its minimal state-space realization (13). Then **OPo** is solvable iff (A, C_2) is detectable. In this case, the optimal K and the corresponding minimal $\|T\|_2^2$ are given by

$$K^{\rm opt} = \mathrm{e}^{-sh}\left[\begin{array}{c|c} A_L & L \\ \hline -C_1 & 0 \end{array}\right] - \pi_h\left\{\left[\begin{array}{c|c} -A_L' & C_2' \\ \hline -C_1Y & 0 \end{array}\right]\right\}\left[\begin{array}{c|c} A_L & L \\ \hline C_2 & \mathrm{I} \end{array}\right]$$

and

$$\|T^{\rm opt}\|_2^2 = \left\|\left[\begin{array}{c|c} A_L & B + LD_2 \\ \hline C_1 & 0 \end{array}\right]\right\|_2^2 - \left\|\pi_h\left\{\left[\begin{array}{c|c} -A_L' & C_2' \\ \hline -C_1Y & 0 \end{array}\right]\right\}\right\|_2^2,$$

where Y is the stabilizing solution of ARE (17) and L is as defined by (16).

It is worth mentioning that the solution in Theorem 2 is based on a standard H^2 algebraic Riccati equation, see [21, §14.5]. In the next subsection we shall see that this is no longer true for the solution of the two-side problem.

B. *The two-side problem*

We are in the position to extend the state-space solution of the one-side problem from the previous subsection to the two-side setting. To this end, consider the following composite system, which is a two-side counterpart of (13) and is given by its minimal state-space realization

$$G = \begin{bmatrix} G_1 & G_3 \\ G_2 & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & 0 & D_3 \\ C_2 & D_2 & 0 \end{array}\right]. \tag{19}$$

In addition to the assumption $\boldsymbol{\mathcal{A}_5}$ we will assume hereafter that

$\boldsymbol{\mathcal{A}_6}$: $D_3'D_3 = I$.

Note that, similarly to $\boldsymbol{\mathcal{A}_5}$, this assumption does not imply any loss of generality.

Since the two-side solution is more complicated then its one-side counterpart and involves nontrivial BDE's, (3), (4), a more detailed state-space structure of (19) is required to carry out the derivations. Bringing in the canonical decomposition of the "A" matrix of (19) with respect to the second input and the second output leads to the following form:

$$G = \left[\begin{array}{ccc|cc} A_3 & A_{12} & A_{13} & B_{11} & B_{12} \\ 0 & A_{1\mathrm{i}} & A_{23} & B_{21} & 0 \\ 0 & 0 & A_2 & B_{31} & 0 \\ \hline C_{11} & C_{12} & C_{13} & 0 & D_3 \\ 0 & 0 & C_{23} & D_2 & 0 \end{array}\right], \tag{20}$$

where the pair (A_3, B_{12}) is controllable and the pair (A_2, C_{23}) is observable. Without loss of generality, we may assume that the realization above has an additional structure:

$$A_2 = \begin{bmatrix} A_{2\mathrm{s}} & 0 \\ 0 & A_{2\mathrm{u}} \end{bmatrix}, \; A_3 = \begin{bmatrix} A_{3\mathrm{u}} & 0 \\ 0 & A_{3\mathrm{s}} \end{bmatrix}, \tag{21}$$

$$A_{12} = \begin{bmatrix} 0 \\ \times \end{bmatrix}, \; A_{13} = \begin{bmatrix} 0 & \times \\ \times & 0 \end{bmatrix}, \; A_{23} = \begin{bmatrix} \times \\ 0 \end{bmatrix}', \tag{22}$$

where $A_{2\mathrm{s}}$ and $A_{3\mathrm{s}}$ are Hurwitz, $A_{2\mathrm{u}}$ and $A_{3\mathrm{u}}$ are anti-stable, and "$\times$" stand for irrelevant blocks. Define matrices E_2 and E_3 through the equalities

$$A_2E_2 = E_2A_{2u} \quad \text{and} \quad E_3'A_3 = A_{3u}E_3'$$

and pick any F_s and L_s such that $A_3 + B_{12}F_sE_3'$ and $A_2 + E_2L_2C_{23}$ are Hurwitz.

To derive the state-space form of Lemma 1, introduce the following pair of constrained Sylvester equations:

$$A_{3\mathrm{u}}Z_2 - Z_2(A_2 - B_{31}D_2'C_{23}) = E_3(A_{13} - B_{11}D_2^+C_{23}), \tag{23a}$$

$$\left(E_3'B_{11} + Z_2B_{31}\right)D_2^{\perp} = 0, \tag{23b}$$

and

$$Z_3A_{2\mathrm{u}} - (A_3 - B_{12}D_3^+C_{11})Z_3 = (A_{13} - B_{12}D_3^+C_{13})E_2, \tag{24a}$$

$$D_3^{\perp}(C_{13}E_2 + C_{11}Z_3) = 0. \tag{24b}$$

Define also

$$J_2 := \begin{bmatrix} E_3' & 0 & Z_2 \end{bmatrix} \quad \text{and} \quad J_3 := \begin{bmatrix} Z_3 \\ 0 \\ E_2 \end{bmatrix},$$

where the partitioning corresponds to that of the "A"-matrix in (20). A straightforward algebra yields then that J_2 and J_3 satisfy the following equalities

$$J_2(A - B_1D_2'C_2) = A_{3u}J_2, \quad J_2B_1D_2^{\perp} = 0, \tag{25}$$

$$(A - B_2D_3'C_1)J_3 = J_3A_{2u}, \quad D_3^{\perp}C_1J_3 = 0. \tag{26}$$

Define also

$$L_t := J_3L_s, \quad F_t := F_sJ_2, \tag{27}$$

$$\begin{bmatrix} \bar K_1 & \bar K_3 \\ \bar K_2 & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A_{FL} & L_k & B_2 \\ \hline F_k & 0 & I \\ C_2 & I & 0 \end{array}\right] \quad \text{and} \quad \begin{bmatrix} \bar G_1 & \bar G_3 \\ \bar G_2 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A_t & B_2(F-F_t) & B_1 + L_t D_2 & B_2 \\ -(L-L_t)C_2 & A_{FL} & -(L-L_t)D_2 & B_2 \\ \hline C_1 + D_3 F_t & D_3(F-F_t) & 0 & D_3 \\ C_2 & -C_2 & D_2 & 0 \end{array}\right] \tag{32}$$

and finally

$$L_k := \begin{bmatrix} -E_3(J_2 B_1 D_2' + Z_2 E_2 L_s) \\ 0 \\ E_2 L_s \end{bmatrix}, \tag{28a}$$

$$F_k := \begin{bmatrix} F_s E_3' & 0 & -(D_3' C_1 J_3 + F_s E_3' Z_3) E_2' \end{bmatrix} \tag{28b}$$

(the partitioning corresponds to that in (20)), which satisfy

$$J_2(L_k + B_1 D_2') = 0 \quad \text{and} \quad (F_k + D_3' C_1) J_3 = 0. \tag{29}$$

The following result is essentially from [5]:

Lemma 3: **OP** is stabilizable iff $A_{1\mathrm{i}}$ is Hurwitz and there exist Z_2 and Z_3 satisfying (23) and (24), respectively. If these conditions hold, then all stabilizing $K \in H^\infty$ and all the corresponding stabilized T's can be characterized by (5) and (6) respectively, where $Q \in H^\infty$ but otherwise arbitrary and

$$\begin{bmatrix} \bar K_1 & \bar K_3 \\ \bar K_2 & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A_k & L_k & B_2 \\ \hline F_k & 0 & I \\ C_2 & I & 0 \end{array}\right], \tag{30}$$

$$\begin{bmatrix} \bar G_1 & \bar G_3 \\ \bar G_2 & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A_t & B_1 + L_t D_2 & B_2 \\ \hline C_1 + D_3 F_t & 0 & D_3 \\ C_2 & D_2 & 0 \end{array}\right], \tag{31}$$

with $A_t = A + B_2 F_t + L_t C_2$, where L_t, F_t are as defined in (27) and $A_k = A + B_2 F_t + L_t C_2$, where L_k, F_k are as defined in (28).

The lemma above is an extension of the state-space solution of the one-side stabilization problem, presented in Lemma 2. We see that, similarly to the one-side stabilization, also in the two-side case the stabilized setting (31) is of the the same order as the original problem, which means that the stabilization procedure can be performed without rising the problem complexity. An important difference between the one and two-side solutions is due to the fact that in the one-side problem the gain L is unconstrained. This property was exploited in the previous subsection to construct a parameterization satisfying (7). On the other hand, in the two-side case the gain matrices F_k and L_k defined by (28) are constrained and so is the matrix A_t of $\bar G_2$ and $\bar G_3$ in (30).

To overcome this difficulty, our next step is to show that the constraints on the gain matrices in (30) can be partially removed, at the expense of the growing order of the resulting stabilized setup. The following result, whose proof is omitted from this paper but can be found in [5], shows that plugging any stabilizing gain matrices satisfying (29) into (30) will still render (5) a complete parameterization of stabilizing solutions.

Lemma 4: Given that **OP** is stabilizable, all stabilizing $K \in H^\infty$ and all corresponding stabilized T's can be characterized by (5) and (6) respectively, where $Q \in H^\infty$ but otherwise arbitrary and $\bar K_i$ and $\bar G_i$, $i \in \{1, 2, 3\}$, are given by (32) on the top of this page, with A_t as in Lemma 3 and $A_{FL} = A + B_2 F + L C_2$, where L, F are any matrices of an appropriate dimensions satisfying

$$J_2(L + B_1 D_2') = 0 \quad \text{and} \quad (F + D_3' C_1) J_3 = 0 \tag{33}$$

and guarantee that A_{FL} is Hurwitz.

Although the choice of the gain matrices F and L in (32) is still restricted by (33), it will be shown below that the constraints imposed by these equalities are loose enough to allow construction of parameterizations that satisfy (7).

Following the line of the one-side solution from the previous subsection, it may seems natural to choose L and F based on standard H^2 ARE's [21, §14.5]. Note, however, that in the two-side case these equations, may be not solvable, since even in stabilizable problems the pairs $(A, C_2)/(A, B_2)$ are not necessarily detectable/stabilizable. This issue can be circumvented by introducing the following modified ARE's with shifted "A"-matrices

$$(A + L_t C_2)' X + X(A + L_t C_2) - (X B_2 + C_1' D_3)(B_2' X + D_3' C_1) + C_1' C_1 = 0 \tag{34}$$

and

$$(A + B_2 F_t) Y + Y(A + B_2 F_t)' - (Y C_2' + B_1 D_2')(C_2 Y + D_2 B_1') + B_1 B_1' = 0. \tag{35}$$

These equations can be considered only if Z_2 and Z_3 involved in the definitions of F_t and L_t are well defined. Namely, only if (23) and (24) are solvable, which, however, is guaranteed for any stabilizable problem. The following statements reveal some properties of ARE's (34) and (35).

Claim 1: Given that **OP** is stabilizable, ARE's (34) and (35) have unique stabilizing solutions. Moreover, these solutions are positive semi-definite.

Proof: Below we prove the statements for the equation (34) only. The proof for the second equation can be constructed using similar arguments. According to [21, Corollary 13.10], to prove the statement it is sufficient to show that the realization

$$\left[\begin{array}{c|c} A + L_t C_2 & B_2 \\ \hline C_1 & D_3 \end{array}\right]$$

has no invariant zeros on the imaginary axis and is stabilizable. To verify the former condition, note that invariant zeros of this realization either coincide with the transmission zeros or with the uncontrollable poles of the realization. By the construction of L_t, the only source of imaginary-axis eigenvalues in $A + L_t C_2$ is the A_3 sub-block, which is controllable from B_2. At the same time, the transmission zeros of the considered system coincide with those of G_3 and can not be located on the imaginary axes due to $\boldsymbol{\mathcal{A}_3}$. The stabilizability of the

$$K^{\rm opt} = {\rm e}^{-sh}\left[\begin{array}{c|c} A_{FL} & L \\ \hline -F & 0\end{array}\right] - \left[\begin{array}{c|c} A_{FL} & B_2 \\ \hline F & {\rm I}\end{array}\right]\pi_h\left\{\left[\begin{array}{cc|c} -(A+B_2F)' & (C_1+D_3F)'C_1Y & -XL \\ 0 & -(A+LC_2)' & -C_2' \\ \hline -B_2' & D_3'C_1Y & 0\end{array}\right]\right\}\left[\begin{array}{c|c} A_{FL} & L \\ \hline C_2 & {\rm I}\end{array}\right], \tag{42}$$

$$\|T^{\rm opt}\|_2^2 = \left\|\left[\begin{array}{cc|c} A_t & B_2(F-F_t) & B_1+L_tD_2 \\ -(L-L_t)C_2 & A_{FL} & -(L-L_t)D_2 \\ \hline C_1+D_3F_t & D_3(F-F_t) & 0\end{array}\right]\right\|_2^2 - \left\|\pi_h\left\{\left[\begin{array}{cc|c} -(A+B_2F)' & (C_1+D_3F)'C_1Y & -XL \\ 0 & -(A+LC_2)' & -C_2' \\ \hline -B_2' & D_3'C_1Y & 0\end{array}\right]\right\}\right\|_2^2, \tag{43}$$

realization follows immediately by the construction of L_t, which completes the proof. ■

Claim 2: Given that **OP** is stabilizable, the stabilizing solutions of (34) and (35) satisfy $XJ_3 = 0$ and $J_2Y = 0$.

Proof: As before, we will consider equation (34) only, since the proof for the second equation can be constructed using similar arguments. Pre- and post-multiplying (34) by J_3' and J_3, respectively, and using (26) yields

$$A_{2u}'J_3'XJ_3 + J_3'XJ_3A_{2u} - J_3'XB_2B_2'XJ_3 + J_3'C_2'L_t'XJ_3 + J_3'XL_tC_2J_3 = 0. \tag{36}$$

Substituting the definitions of L_t, F_t and reorganizing the equation above, we get

$$(A_{2u} + L_sC_{23}E_2)'J_3'XJ_3 + J_3'XJ_3(A_{2u} + L_sC_{23}E_2) = J_3'XB_2B_2'XJ_3. \tag{37}$$

By the construction of L_s, the matrix $A_{2u} + L_sC_{23}E_2$ is Hurwitz. Considering the equality above as a Lyapunov equation in $J_3'XJ_3$, which is constrained to be positive semi-definite, and taking into account that the right-hand side of the equality is positive semi-definite as well, implies that $J_3'XJ_3 = 0$ and, as a result, $XJ_3 = 0$. ■

The result of Claim 2 implies that the stabilizing solutions of (34) and (35) satisfy $XL_tC_2 = XJ_3L_sC_2 = 0$ and $B_2F_tY = B_2F_sJ_2Y = 0$, which, in turn, implies that they satisfy also the standard H^2 Riccati equations. Note, however, that stabilizing solutions of the shifted equations are not necessarily stabilizing for the standard ARE's. The relation between the two-side H^2 optimization problem with stability constraints and non stabilizing solutions of standard ARE's was perceived earlier in [8], were a problem without preview was studied and the notion of semi-stabilizing ARE solution was introduced. In the current work, we use the shifted Riccati equations as an alternative to the definition of semi-stabilizing solutions. This is convenient in the context of numerical implementation, since shifted equations can be solved using the standard routines for computing the stabilizing ARE solution.

At this point we can choose the stabilizing gain matrices for (32) as

$$F = -B_2'X - D_3'C_1 \quad \text{and} \quad L = -YC_2' - B_1D_2', \tag{38}$$

where X and Y are the stabilizing solutions of (34) and (35) respectively. It can be shown by straightforward algebra that for this choice of L and F the matrix A_{FL} is Hurwitz. Note also that the result of Claim 2 together with (38) imply that

$$(F + D_3'C_1)J_3 = -B_2'XJ_3 = 0, \tag{39}$$

$$J_2(L + B_1D_2') = -J_3Yc_2' = 0, \tag{40}$$

showing that the choice of F and L, in (38) satisfies the conditions of Lemma 4. Finally, it can be shown by straightforward, yet tedious, calculations that for this choice of the gain matrices the conditions in (7) hold and

$$\bar G_3^{\sim}\bar G_1\bar G_2^{\sim} = \left[\begin{array}{cc|c} -(A+B_2F)' & XB_1(B_1+LD_2)' & XB_1D_2' \\ 0 & -(A+LC_2)' & -C_2' \\ \hline -B_2' & -FY & 0\end{array}\right] = \left[\begin{array}{cc|c} -(A+B_2F)' & (C_1+D_3F)'C_1Y & -XL \\ 0 & -(A+LC_2)' & -C_2' \\ \hline -B_2' & D_3'C_1Y & 0\end{array}\right]. \tag{41}$$

At this point, substituting (30) and (41) into (10), (11) yields the main result of the current paper.

Theorem 3: Let G be given by its minimal state-space realization (19), having the structure specified by (20), (21). Then **OP** is solvable iff A_{1i} is Hurwitz and there exist Z_2 and Z_3 satisfying (23) and (24) respectively. If these conditions hold, then the optimal solution and the corresponding minimal $\|T\|_2^2$ are given by (42) and (43), respectively, where A_t is as defined in Lemma 3, X, Y are stabilizing solutions of ARE's (34), (35) and $A_{\rm FL} = A + B_2F + LC_2$ with L, F as defined in (38).

Theorem 3 concludes the work, providing an explicit and numerically feasible solution of **OP**. An important difference between the one- and two-side solution is that the latter relies on the modified ARE's, with A matrices shifted by terms obtained from the solution of the constrained Sylvester equations (23) and (24). The computation of these terms can be complicated by the fact that (23a) and (24a) may have multiple solutions. It may be shown however that, taking into account the constraints (23b) and (24b), Z_2 and Z_3 are unique once they exist and there exists an efficient method for their calculation, see [3], [5] for more details.

IV. Concluding remarks

In this paper we have studied the H^2 two-side model matching problem, which can be considered as a unified setting for control and estimation problems with preview and asymptotic behavior constraints. As a first step, recent results from [3], [5] are used to resolve asymptotic behavior constraints and to reduce the problem to an unconstrained L^2 optimization problem. Then, the optimization problem is solved using standard Hilbert space arguments. This procedure results in explicit state-space formulae for the optimal solution and for the optimal achievable norm. The solution relies upon two constrained algebraic Sylvester equations, associated with asymptotic behavior, and two algebraic Riccati equations (ARE's), which do not depend on the length of preview.

In the one-side case, the ARE involved in the solution is not affected by asymptotic behavior constraints and is actually the standard H^2 Riccati equation. In the two-side setting, however, the situation is more complicated, since even in stabilizable problems the standard Riccati equations might have no stabilizing solutions. In this work, the two-side problem was solved in terms of stabilizing solutions of modified AREs with "A" matrices shifted by terms obtained from the solutions of the Sylvester equations. This can be considered as an alternative to the concept of a semi-stabilizing solution of ARE proposed in [8].

Practical merits of the results presented in this paper are demonstrated in [5] by means of laboratory experiments.

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