Pseudorational Behaviors and Bezoutians

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Abstract— **Behavioral system theory has been successful in providing a viewpoint that does not depend on a priori notions of inputs/outputs. While there are some attempts to extend this theory to infinite-dimensional systems, for example, delay systems, the overall picture seems to remain still incomplete.**

The first author has studied a class of infinite-dimensional systems called pseudorational. This class allows a compact fractional representation for systems having bounded-time memory. It is particularly appropriate for extending the behavioral framework to infinite-dimensional context.

We have recently studied several attempts to extend this framework to a behavioral context. Among them are characterizations of behavioral controllability, particularly involving a coprimeness condition over an algebra of distributions, and some stability tests involving Lyapunov functions derived from Bézoutians.

This article gives a brief overview of pseudorational transfer functions, controllability issues and related criteria, path integrals, and finally the connection with Lyapunov functions derived from Bézoutians.

I. Introduction

Behavioral system theory has become a successful framework in providing a viewpoint that does not depend on a priori notions of inputs/outputs. An introductory and tutorial account is given in [7], [2]. In particular, this theory successfully provides such notions as controllability, without an explicit reference to state space formalism. One also obtains several interesting and illuminating consequences of controllability, for example, direct sum decomposition of the signal space with a controllable behavior $\mathscr{B}$ as a direct summand.

There are some attempts to extend this theory to infinite-dimensional systems, for example, delay systems; some rank conditions for behavioral controllability have been obtained; see, e.g., [1], [3], [6]. While these results give a nice generalization of their finite-dimensional counterparts, the overall picture still needs to be further studied in a more general and perhaps abstract setting. For example, one wants to see how the notion of zeros and poles can affect controllability in an abstract setting. This is to some extent accomplished in [3], [1], but we here intend to give a theory in a more general, and unified setting, and provide a framework in a well-behaved class of infinite-dimensional systems called pseudorational.

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In [9], [11], the first author introduced the notion of pseudorational impulse responses. Roughly speaking, an impulse response is said to be pseudorational if it is expressible as a ratio of distributions with compact support, e.g., $G = p^{-1} * q$ (Warning: we used $q^{-1} * p$ in [9], [11]) This leads to an input/output relation

$$p * y = q * u, \tag{1}$$

and various system properties have been studied associated to it: for example,

1) realization procedure
2) complete characterization of spectra in terms of the denominator of the transfer function
3) stability characterization in terms of the spectrum location
4) relations between controllability and coprimeness conditions.

The representation (1) is also suitable for behavioral study. In this paper, we survey results obtained mainly in [14], [15]:

- behavioral controllability and its characterization in terms of coprimeness of the pair (p,q),
- Lyapunov stability and related conditions derived from Bézoutians, and finally give indications on
- dissipation conditions.

II. Notation and Nomenclature

$\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ ($\mathscr{C}^\infty$ for short) is the space of C^∞ functions on $(-\infty,\infty)$. Similarly for $\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ with higher dimensional codomains. $\mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ denote the space of $\mathbb{R}^{\mathtt{q}}$-valued C^∞ functions having compact support in $(-\infty,\infty)$. $\mathscr{D}'(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ is its dual, the space of distributions. $\mathscr{D}'_+(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ is the subspace of $\mathscr{D}'$ with support bounded on the left. $\mathscr{E}'(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ denotes the space of distributions with compact support in $(-\infty,\infty)$. $\mathscr{E}'(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ is a convolution algebra and acts on $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ by the action: $p* : \mathscr{C}^\infty(\mathbb{R},\mathbb{R}) \to \mathscr{C}^\infty(\mathbb{R},\mathbb{R}) : w \mapsto p * w$. $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ is a module over $\mathscr{E}'$ via this action. Similarly, $\mathscr{E}'(\mathbb{R}^2,\mathbb{R}^{\mathtt{q}})$ denotes the space of distributions in two variables having compact support in $\mathbb{R}^2$. For simplicity of notation, we may drop the range space $\mathbb{R}^{\mathtt{q}}$ and write $\mathscr{E}'(\mathbb{R})$, etc., when no confusion is likely,

A distribution α is said to be of *order at most m* if it can be extended as a continuous linear functional on the space of m-times continuously differentiable functions. Such a distribution is said to be of *finite order*. The largest number m, if one exists, is called the *order* of α ([4], [5]). The delta distribution δ_a ($a \in \mathbb{R}$) is of order zero, while its derivative δ'_a is of order one, etc. A distribution with compact support is known to be always of finite order ([4], [5]).

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The Laplace transform of $p \in \mathscr{E}'(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ is defined by

$$\mathscr{L}[p](\zeta) = \hat{p}(\zeta) := \langle p, e^{-\zeta t}\rangle_t \tag{2}$$

where the action is taken with respect to t. Likewise, for $p \in \mathscr{E}'(\mathbb{R}^2,\mathbb{R}^{\mathtt{q}})$, its Laplace transform is defined by

$$\hat{p}(\zeta,\eta) := \langle p, e^{-(\zeta s+\eta t)}\rangle_{s,t} \tag{3}$$

where the distribution action is taken with respect to two variables s and t. For example, $\mathscr{L}[\delta_s'' \otimes \delta_t'] = \zeta^2 \cdot \eta$.

By the well-known Paley-Wiener theorem [4], [5], $\hat{p}(\zeta)$ is an entire function of exponential type satisfying the *Paley-Wiener estimate*

$$|\hat{p}(\zeta)| \le C(1+|\zeta|)^r e^{a|\mathrm{Re}\,\zeta|} \tag{4}$$

for some $C, a \ge 0$ and a nonnegative integer r.

Likewise, for $p \in \mathscr{E}'(\mathbb{R}^2,\mathbb{R}^{\mathtt{q}})$, there exist $C, a \ge 0$ and a nonnegative integer r such that its Laplace transform

$$|\hat{p}(\zeta,\eta)| \le C(1+|\zeta|+|\eta|)^r e^{a(|\mathrm{Re}\,\zeta|+|\mathrm{Re}\,\eta|)}. \tag{5}$$

This is also a sufficient condition for a function $\hat{p}(\cdot,\cdot)$ to be the Laplace transform of a distribution in $\mathscr{E}'(\mathbb{R}^2,\mathbb{R}^{\mathtt{q}})$. We denote by $\mathscr{PW}$ the class of functions satisfying the estimate above for some C, a, m. In other words, $\mathscr{PW} = \mathscr{L}[\mathscr{E}']$.

Other spaces, such as L^2, L^2_{loc} are all standard. For a vector space X, $X^{\mathtt{n}}$ and $X^{\mathtt{n}\times\mathtt{m}}$ denote, respectively, the spaces of $\mathtt{n}$ products of X and the space of $\mathtt{n}\times\mathtt{m}$ matrices with entries in X. When a specific dimension is immaterial, we will simply write $X^{\bullet}$, $X^{\bullet\times\bullet}$, etc.

III. Pseudorational Behaviors

We review a few rudiments of pseudorational behaviors as given in [13], [14].

Definition 3.1: Let R be an $\mathtt{p}\times\mathtt{w}$ matrix ($\mathtt{w} \ge \mathtt{p}$) with entries in $\mathscr{E}'$. It is said to be *pseudorational* if there exists a $\mathtt{p}\times\mathtt{p}$ submatrix P such that

1) $P^{-1} \in \mathscr{D}'_+(\mathbb{R})$ exists with respect to convolution;
2) $\mathrm{ord}(\det P^{-1}) = -\mathrm{ord}(\det P)$, where $\mathrm{ord}\,\psi$ denotes the order of a distribution ψ [4], [5] (for a definition, see the Appendix).

Definition 3.2: Let R be pseudorational as defined above. The *behavior* $\mathscr{B}$ defined by R is given by

$$\mathscr{B} := \{w \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}}) : R * w = 0\} \tag{6}$$

The convolution $R*w$ is taken in the sense of distributions. Since R has compact support, this convolution is always well defined [4]. The *distributional behavior* $\mathscr{B}_{\mathscr{D}'}$ defined by R is given by

$$\mathscr{B}_{\mathscr{D}'} := \{w \in (\mathscr{D}')^{\mathtt{w}} | R * w = 0\}. \tag{7}$$

where $\mathscr{D}'$ denotes the space of distributions on $\mathbb{R}$.

Remark 3.3: We here took $\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ as the signal space in place of $L^2_{loc}(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ in [14], but the basic structure remains intact.

A state space formalism is possible for this class and it yields various nice properties as follows:

Suppose, without loss of generality, that R is partitioned as $R = \begin{bmatrix} P & Q \end{bmatrix}$ such that P satisfies the invertibility condition of Definition 3.1, i.e., we consider the kernel representation

$$P*y + Q*u = 0 \tag{8}$$

where $w := \begin{bmatrix} y & u \end{bmatrix}^T$ is partitioned conformably with the sizes of P and Q.

A nice consequence of pseudorationality is that this space X is always a closed subspace of the following more tractable space X^P:

$$X^P := \{x \in (L^2_{[0,\infty)})^{\mathtt{p}} \,|\, P*x|_{[0,\infty)} = 0\}, \tag{9}$$

and it is possible to give a realization using X^P as a state space. The state transition is generated by the left shift semigroup:

$$(\sigma_\tau x)(t) := x(t+\tau)$$

and its infinitesimal generator A determines the spectrum of the system ([9]). We have the following facts concerning the spectrum, stability, and coprimeness of the representation $\begin{bmatrix} P & Q \end{bmatrix}$ ([9], [11], [12], [13]):

Facts 3.4:

1) The spectrum $\sigma(A)$ is given by

$$\sigma(A) = \{\lambda \,|\, \det \hat{P}(\lambda) = 0\}. \tag{10}$$

Furthermore, every $\lambda \in \sigma(A)$ is an eigenvalue with finite multiplicity. The corresponding eigenfunction for $\lambda \in \sigma(A)$ is given by $e^{\lambda t}v$ where $\hat{P}(\lambda)v = 0$. Similarly for generalized eigenfunctions such as $te^{\lambda t}v'$.

2) The semigroup σ_t is exponentially stable, i.e., satisfies for some $C, \beta > 0$

$$\|\sigma_t\| \le Ce^{-\beta t}, \quad t \ge 0,$$

if and only if there exists $\rho > 0$ such that

$$\sup\{\mathrm{Re}\,\lambda : \det \hat{P}(\lambda) = 0\} \le -\rho.$$

As a rather direct consequence, we have the following lemma on stability of behaviors. Let $R \in (\mathscr{E}'(\mathbb{R},\mathbb{R}^{\mathtt{q}}))^{\mathtt{p}\times\mathtt{q}}$ be pseudorational, and let $\mathscr{B}$ be the autonomous behavior defined by R, i.e.,

$$\mathscr{B} = \{w : R*w = 0\}. \tag{11}$$

We discuss stability conditions in terms of R.

Lemma 3.5: The behavior $\mathscr{B}$ is exponentially stable if and only if

$$\sup\{\mathrm{Re}\,\lambda : \det \hat{R}(\lambda) = 0\} < 0. \tag{12}$$

Proof See [15]. □

IV. Controllability and Coprimeness

We now introduce the notion of controllability [2] in the present context.

Definition 4.1: Let R be pseudorational, and $\mathscr{B}$ the behavior associated to it. $\mathscr{B}$ is said to be *controllable* if for every pair $w_1, w_2 \in \mathscr{B}$, there exists $T \ge 0$ and $w \in \mathscr{B}$, such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t-T)$ for $t \ge T$

In other words, every pair of trajectories can be concatenated into one trajectory that agrees with them in the past and future.

We also introduce an extended notion of controllability as follows:

Definition 4.2: Let R be pseudorational, and $\mathscr{B}_{\mathscr{D}'}$ be the distributional behavior (7). $\mathscr{B}_{\mathscr{D}'}$ is said to be *distributionally controllable* if for every pair $w_1, w_2 \in \mathscr{B}$, there exists $T \geq 0$ and $w \in \mathscr{B}$, such that $w|_{(-\infty,0)} = w_1$ on $(-\infty, 0)$, and $w|_{(T,\infty)} = \sigma_{-T} w_2$ on (T, ∞).

Let us now introduce various notions of coprimeness.

Definition 4.3: The pair (P, Q), $P, Q \in \mathscr{E}'(\mathbb{R})$ is said to be *spectrally coprime* if $\hat{P}(s)$ and $\hat{Q}(s)$ have no common zeros. It is *approximately coprime* if there exist sequences $\Phi_n, \Psi_n \in \mathscr{E}'(\mathbb{R})$ such that $P * \Phi_n + Q * \Psi_n \to \delta I$ in $\mathscr{E}'(\mathbb{R})$. The pair (P, Q) is said to satisfy the *Bézout identity* (or simply *Bézout*), if there exists $\Phi, \Psi \in \mathscr{E}'(\mathbb{R})$ such that

$$P * \Phi + Q * \Psi = \delta I, \tag{13}$$

Or equivalently,

$$\hat{P}(s)\hat{\Phi}(s) + \hat{Q}(s)\hat{\Psi}(s) = I \tag{14}$$

for some entire functions $\hat{\Phi}, \hat{\Psi}$ satisfying the Paley-Wiener estimate (4).

It is well known [2] that controllability admits various nice characterizations in terms of coprimeness, image representation, full rank conditions, etc. We here attempt to give a generalization of such results to the present context. To this end, we confine ourselves to the simplest scalar case, i.e., $p = m = 1$. We will also assume that q also satisfies the condition that the zeros of $\hat{q}(s)$ is contained in a half plane $\{s \,|\, \mathrm{Re}\, s < c\}$ for some $c \in \mathbb{R}$.

Theorem 4.4: Let R be pseudorational, and suppose without loss of generality that R is of form $R := \begin{bmatrix} p & q \end{bmatrix}$ where p satisfies the invertibility condition in Definition 3.1. Let $\mathscr{B}_{\mathscr{D}'}$ be the distributional behavior (7). Then the following statements are equivalent:

1) $\mathscr{B}_{\mathscr{D}'}$ is controllable.
2) There exist $\psi, \phi \in \mathscr{E}'(\mathbb{R})$ such that $p * \phi + q * \psi = \delta$.
3) $\mathscr{B}_{\mathscr{D}'}$ admits an image representation, i.e., there exists M over $\mathscr{E}'(\mathbb{R})$ such that for every $w \in \mathscr{B}_{\mathscr{D}'}$, there exists $\ell \in C^\infty(\mathbb{R})$ such that $w = M * \ell$.
4) $\mathscr{B}_{\mathscr{D}'}$ is a direct summand of $\mathscr{D}'$, i.e., there exists an distributional behavior $\mathscr{B}'$ such that $\mathscr{D}' = \mathscr{B}_{\mathscr{D}'} \oplus \mathscr{B}'$.
5) Let $\Lambda := \{\lambda \in \mathbb{C} | \hat{p}(\lambda) = 0\}$. Suppose that the algebraic multiplicity of each zero $\lambda \in \Lambda$ is globally bounded. There exist $k \geq 0$ and $c > 0$ such that

$$|\lambda^k \hat{q}(\lambda)| \geq c, \quad \forall \lambda \in \Lambda. \tag{15}$$

Proof See [14]. □

V. Path Integrals

The integral

$$\int_{t_1}^{t_2} Q_\Phi(w) dt \tag{16}$$

(or briefly $\int Q_\Phi$) is said to be *independent of path*, or simply a *path integral* if it depends only on the values taken on by w and its derivatives at end points t_1 and t_2 (*but not on the intermediate trajectories between them*).

The following theorem gives equivalent conditions for Φ to give rise to a path integral.

Theorem 5.1: Let $\Phi \in \mathscr{E}'(\mathbb{R}^2)^{\mathtt{q} \times \mathtt{q}}$, and Q_Φ the quadratic differential form associated with Φ. The following conditions are equivalent:

(i) $\int Q_\Phi$ is a path integral;
(ii) $\partial \Phi = 0$;
(iii) $\int_{-\infty}^{\infty} Q_\Phi(w) dt = 0$ for all $w \in \mathscr{D}(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$;
(iv) the expression $\hat{\Phi}(\zeta, \eta)/(\zeta + \eta)$ belongs to the class $\mathscr{PW}$.
(v) there exists a two-variable matrix $\Psi \in \mathscr{E}'(\mathbb{R}^2)^{\mathtt{q} \times \mathtt{q}}$ that defines a Hermitian bilinear form on $(\mathscr{C}^\infty)^{\mathtt{q}} \otimes (\mathscr{C}^\infty)^{\mathtt{q}}$ such that

$$\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w) \tag{17}$$

for all $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$.

For a proof, see [15].

VI. Path Integrals along a Behavior

Generalizing the results of Section V on path integrals in the unconstrained case, we now study path integrals *along a behavior* $\mathscr{B}$.

Definition 6.1: Let $\mathscr{B}$ be the behavior (6) with pseudorational R. The integral $\int Q_\Phi$ is said to be *independent of path* or a *path integral along* $\mathscr{B}$ if the path independence condition holds for all $w_1, w_2 \in \mathscr{B}$.

Let $\mathscr{B}$ be as above. We assume that $\mathscr{B}$ also admits an *image representation*, i.e., $\mathscr{B} = M * \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$. This implies that $\mathscr{B}$ is controllable. In fact, for a polynomial R, controllability of $\mathscr{B}$ is also sufficient for the existence of an image representation, but in the present situation, it is not fully known. A partial necessary and sufficient result for the scalar case is given in [14].

We then have the following theorem.

Theorem 6.2: Let $\mathscr{B}$ be a behavior defined by a pseudorational R, and suppose that $\mathscr{B}$ admits an image representation $\mathscr{B} = \mathrm{im} M *$. Let Φ be as above. Then the following conditions are equivalent:

(i) $\int Q_\Phi$ is a path integral along $\mathscr{B}$;
(ii) there exists $\Psi = \Psi^* \in \mathscr{PW}^{q \times q}[\zeta, \eta]$ such that

$$\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w) \tag{18}$$

for all $w \in \mathscr{B}$;
(iii) $\int Q_{\Phi'}$ is a path integral where Φ' is defined by $\Phi'(\zeta, \eta) := M^T(\zeta)\Phi(\zeta, \eta)M(\eta)$;
(iv) $\partial \Phi' = 0$;
(v) there exists $\Psi' = (\Psi')^\bullet = \mathscr{PW}^{q \times q}[\zeta, \eta]$ such that

$$\frac{d}{dt} Q_{\Psi'}(\ell) = Q_{\Phi'}(\ell)$$

for all $\ell \in \mathscr{C}^\infty$, i.e., $\Psi'^\bullet = \Phi'$.

For a proof, see [15].

VII. Lyapunov stability

A characteristic feature in stability for the class of pseudorational transfer functions is that asymptotic stability is determined by the location of poles, i.e., zeros of $\det\hat{R}(\zeta)$. Indeed, as we have seen in Lemma 3.5, the behavior

$$\mathscr{B} = \{w : R * w = 0\},$$

is exponentially stable if and only if $\sup\{\mathrm{Re}\,\lambda : \det\hat{R}(\lambda) = 0\} < 0$, and this is determined how each characteristic solution $e^{\lambda t}a$, $a \in \mathbb{C}^{\mathrm{q}}$ ($\det\hat{R}(\lambda) = 0$), behaves. This plays a crucial role in discussing stability in the Lyapunov theory. We start with the following lemma which tells us how $p \in \mathscr{E}'(\mathbb{R}, \mathbb{R}^{\mathrm{q}})$ acts on $e^{\lambda t}$ via convolution:

We give some preliminary notions on positivity (resp. negativity).

Definition 7.1: The QDF Q_Φ induced by Φ is said to be *nonnegative* (denoted $Q_\Phi \geq 0$) if $Q_\Phi(w) \geq 0$ for all $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{q}})$, and *positive* (denoted $Q_\Phi(w) > 0$) if it is nonnegative and $Q_\Phi(w) = 0$ implies $w = 0$.

Let $\mathscr{B} = \{w : R * w = 0\}$ be a pseudorational behavior. The QDF Q_Φ induced by Φ is said to be *$\mathscr{B}$-nonnegative* (denoted $Q_\Phi \overset{\mathscr{B}}{\geq} 0$) if $Q_\Phi(w) \geq 0$ for all $w \in \mathscr{B}$, and *$\mathscr{B}$-positive* (denoted $Q_\Phi(w) \overset{\mathscr{B}}{>} 0$) if it is $\mathscr{B}$-nonnegative and if $Q_\Phi(w)$ and $w \in \mathscr{B}$ imply $w = 0$. $\mathscr{B}$-nonpositivity and $\mathscr{B}$-negativity are defined if the respective conditions hold for $-Q_\Phi$.

We say that Q_Φ *weakly strictly positive along* $\mathscr{B}$ if

- Q_Φ is $\mathscr{B}$-positive; and
- for every $\gamma > 0$ there exists c_γ such that $\overline{a}^T\hat{\Phi}(\overline{\lambda}, \lambda)a \geq c_\gamma\|a\|^2$ for all λ with $\hat{p}(\lambda) = 0$, $\mathrm{Re}\,\lambda \geq -\gamma$ and $a \in \mathbb{C}^q$.

Similarly for *weakly strict negativity along* $\mathscr{B}$.

For a polynomial $\hat{\Phi}$, $\mathscr{B}$-positivity clearly implies the second condition. However, for pseudorational behaviors, this may not be true. Note that we require the above estimate only for the eigenvalues λ, whence the term “weakly”.

Theorem 7.2: Let $\mathscr{B}$ be as above. $\mathscr{B}$ is asymptotically stable if there exists $\Psi = \Psi^* \in \mathscr{E}'(\mathbb{R}^2)^{\mathrm{q}\times\mathrm{q}}$ whose elements are measures (i.e., distributions of order 0) such that Q_Ψ is weakly strictly positive along $\mathscr{B}$ and $\overset{\bullet}{\Psi}$ weakly strictly negative along $\mathscr{B}$.

Proof See [15]. □

VIII. The Bézoutian

We have seen that exponential stability can be deduced from the existence of a suitable positive definite quadratic form Ψ that works as a Lyapunov function. The question then hinges upon how one can find such a Ψ. The objective of this section is to show that for the single-variable case, the Bézoutian gives a universal construction for obtaining a Lyapunov function.

In this section we confine ourselves to the case $q = 1$, that is, given $p \in \mathscr{E}'$, we consider the behavior

$$\mathscr{B} = \{w : p * w = 0\}.$$

Define the Bézoutian $b(\zeta, \eta)$ by

$$b(\zeta, \eta) := \frac{p(\zeta)p(\eta) - p(-\zeta)p(-\eta)}{\zeta + \eta}. \tag{19}$$

Note that this expression belongs to the class $\mathscr{PW}[\zeta, \eta]$, and hence its inverse Laplace transform is a distribution having compact support. Let us further assume that p is a measure, i.e., distribution of order 0. If not, $\hat{p}(s)$ possess (stable) zeros, and we can reduce $\hat{p}(s)$ to a measure by extracting such zeros. For details, see [10].

We have the following theorem:

Theorem 8.1: Suppose that $p \in \mathscr{E}'$ is a measure. The following conditions are equivalent:

(i) $\mathscr{B} = \{w : p * w = 0\}$ is exponentially stable;
(ii) there exists $\rho > 0$ such that $\sup\{\lambda : \hat{p}(\lambda) = 0\} \leq -\rho$;
(iii) $Q_b \geq 0$ and the pair $(p, p^\sim)$ is coprime in the following sense: there exists $\phi, \psi \in \mathscr{E}'$ such that

$$p * \phi + p^\sim * \psi = \delta \tag{20}$$

(iv) Q_b is weakly strictly positive definite, and $Q_{\overset{\bullet}{b}}$ is weakly strictly negative definite.

Proof Omitted. See [15]. □

IX. Disspativity

Definition 9.1: Let $\Phi, \Psi, \Delta \in \mathscr{E}(\mathbb{R}^2)^{q\times q}$. A quadratic differential form Q_Ψ is a *storage function* for Φ if

$$\frac{d}{dt}Q_\Psi \leq Q_\Phi.$$

A quadratic differential form Q_Δ is a *disspation function* if

$$\Delta \geq 0 \text{ and } \int Q_\Phi = \int Q_\Delta.$$

For pseudorational behaviors, we have the following extension of the finite-dimensional case given in [8]:

Theorem 9.2: Let $\Phi \in \mathscr{E}(\mathbb{R}^2)^{q\times q}$. The following conditions are equivalent:

1) For every $w \in \mathscr{D}(\mathbb{R}, \mathbb{R}^q)$,

$$\int Q_\Phi(w)dt \geq 0.$$

2) There exists a storage function for Φ.
3) There exists a dissipation function for Φ.

Proof Omitted. See [16] □

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